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Beyond fuzzy spheres

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Abstract
We study polynomial deformations of the fuzzy sphere, specifically given by the cubic or the Higgs algebra. We derive the Higgs algebra by quantizing the Poisson structure on a surface in $\mathbb{R}^3$. We find that several surfaces, differing by constants, are described by the Higgs algebra at the fuzzy level. Some of these surfaces have a singularity and we overcome this by quantizing this manifold using coherent states for this nonlinear algebra. This is seen in the measure constructed from these coherent states. We also find the star product for this non-commutative algebra as a first step in constructing field theories on such fuzzy spaces.

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1. Introduction

Field theories on fuzzy spheres have been actively pursued in the past few years [1–6]. The primary interest in studying fuzzy spaces stems from the attractive discretization it offers to regularize quantum field theories preserving symmetries. It is known that a lattice discretization, though helpful in regularizing the field theory, breaks the symmetries of the theory under consideration and the full symmetry is realized only when the regulator is taken to zero. This problem does not occur in the fuzzy case. In the process the fermion doubling problem is also avoided [7–9]. Also the possibility of new phases in the continuum theory which breaks translation as well as other global symmetries can be studied through simulations [10–14]. Furthermore, one can incorporate supersymmetry in a precise manner. All these issues and many more attractive features can be found in detail in [6].

Fuzzy spheres and their generalizations were also considered in a Kaluza–Klein framework as extra-dimensional space. There is also a proposal for dynamically generating such spaces in the same framework [5, 15, 16]. Also analytically and through simulations one can study the evolution of geometries and their transitions. Such spaces also arise as backgrounds in string theory with appropriate Chern–Simons coupling [17, 18]. Our considerations here will be applicable in those scenarios too.

In the present work we go beyond the usual Lie algebra characterizing the fuzzy sphere:

$$[X_j, X_k] = \frac{i\alpha\epsilon_{jkl}}{\sqrt{N(N+2)}}X^l, \qquad \sum_i X_i^2 = \alpha^2. \tag{1}$$

What we have in mind is the study of various aspects of a surface whose coordinates satisfy the cubic algebra, also known as the Higgs algebra (HA):

$$[X_+, X_-] = C_1 Z + C_2 Z^3, \qquad [X_\pm, Z] = \pm X_\pm. \tag{2}$$

This algebra originally arose as a symmetry algebra in the study of the 'Kepler problem' in curved spaces, particularly on a sphere [19, 20]. Quantum mechanical Hamiltonian of a particle in a Kepler potential on the surface of a sphere has a dynamical symmetry given by the above algebra and can be used to solve the problem exactly.

The HA can be studied as a deformation of $SU(2)$ or $SU(1,1)$. It can also be considered as a deformation of $SU_q(2)$ [21]. In this sense the HA sits between the Lie and q deformed algebras. Such algebras are not only interesting from the physical point of view but also widely studied in mathematics [22].

In this paper we derive the Higgs algebra by quantizing the Poisson structure on a manifold, which is embedded in $\mathbb{R}^3$. We call this manifold, the Higgs manifold, $\mathcal{M}_H$, as the algebra got from the Poisson algebra is the Higgs algebra. Arlind *et al* [23] also produced new nonlinear deformations of the $SU(2)$ algebra of different kinds which gave them torus geometry as well as topology change. Another algebra that exhibits topology change is the Sklyanin algebra [24].

This paper is organized as follows. Section 2 shows the derivation of the Higgs algebra from the Poisson algebra on the Higgs manifold. The question of topology change in surfaces that are not round spheres is also presented. In this process of topology change, we encounter the singularity which is discussed in detail. In section 3, we briefly introduce the interesting aspects of the HA and focus on its finite-dimensional representations. Section 4 gives the construction of the coherent states (CS) of the HA in detail. Herein, we also provide the measure required for the resolution of unity. These CS are then used in section 5 to obtain the star product. Our conclusions and outlook are presented in section 6.

2. The Higgs manifold

We consider the following embedding in $\mathbb{R}^3$:

$$x^2 + y^2 + (z^2 - \mu)^2 = 1. \tag{3}$$

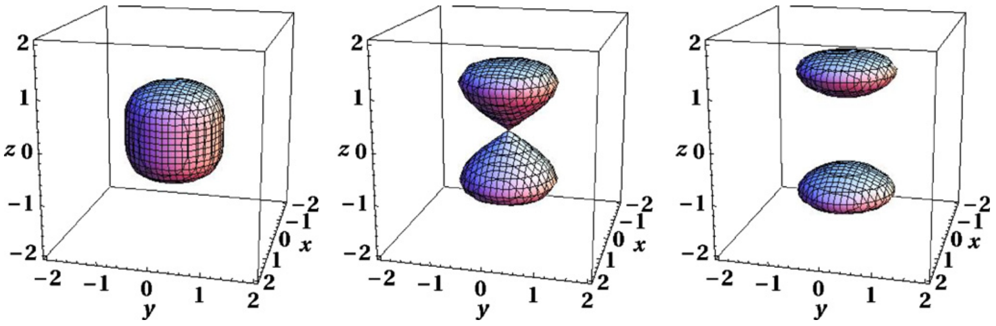
This is the surface we call the Higgs manifold, $\mathcal{M}_H$. μ is a parameter which can be varied. We now analyze this equation for different values of μ.

For $\mu = 1$, it is easy to see that there is a singular point $(x = y = z = 0)$ where the surface degenerates. But in the discrete case, the representations do not display any difficulty at this value. When $\mu < -1$, there are no solutions. For $-1 < \mu < 1$ we have a deformed sphere, but still symmetric under rotations about the z-axis. The surface becomes two disconnected spheres for $\mu > 1$. These are explicitly shown in figure 1 for specific values of μ.

2.1. The singularity

We use cylindrical coordinates to show the conical singularity arising at $\mu = 1$ and $x = y = z = 0$ as shown in figure 1(*b*). The equation of the surface becomes

$$r^2 + (z^2 - 1)^2 = 1. \tag{4}$$



(*a*) $\mu = 0$ (*b*) $\mu = 1$ (*c*) $\mu = 1.5$

Figure 1. Surface plots depicting the change in topology.
(This figure is in colour only in the electronic version)

This acts as a constraint giving z in terms of r. Substituting this in the line element of Euclidean 3-space, in the cylindrical coordinates, $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 + \mathrm{d}z^2$, we get the induced metric on this surface.

$$\mathrm{d}s^2 = \left[1 + \frac{r^2}{4(1-r^2)[1-\sqrt{1-r^2}]}\right]\mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2. \tag{5}$$

As $r \to 0$, the first term of the metric approaches $\frac{3}{2}$. This implies a scaling of r by $\sqrt{\frac{3}{2}}$. This in turn induces a scaling of ϕ by $\sqrt{\frac{2}{3}}$. Thus, the new ϕ coordinate has a range from 0 to $2\pi\left(1 - \sqrt{\frac{2}{3}}\right)$, making the origin a conical singularity. This singularity cannot be removed by a coordinate transformation.

2.2. The Poisson algebra on $\mathcal{M}_H$

We use the Poisson structure on $\mathbb{R}^3$ as defined by [23] to derive the Higgs algebra. This Poisson bracket is given by

$$\{f, g\} = \frac{\partial(C, f, g)}{\partial(x, y, z)}, \tag{6}$$

where $C = x^2 + y^2 + (z^2 - \mu)^2$ and f and g are two functions on $\mathbb{R}^3$.

Using this we find the following:

$$\{x, y\} = 4z^3 - 4\mu z, \tag{7}$$

$$\{z, x\} = 2y, \tag{8}$$

$$\{y, z\} = 2x. \tag{9}$$

This, when quantized, gives the Higgs algebra.

3. The Higgs algebra and its representation

We have shown in the previous section how the cubic Poisson bracket can be induced by a surface that is quartic in z. When we quantize this nonlinear bracket we get the HA. The interest in studying these nonlinear algebras, apart from the physical applications [19, 20], is that we can construct unitary finite- or infinite-dimensional representations. These and other interesting aspects were studied for many of these nonlinear structures, collectively called the polynomial algebras, by various authors [25–30]. In what follows we will explicitly state the representations of importance to us.

Let X_+, X_-, Z be the generators of a three-dimensional polynomial algebra. This algebra is defined by the following commutation relations:

$$[X_+, X_-] = C_1 Z + C_2 Z^3 \equiv f(Z), \qquad [Z, X_\pm] = \pm X_\pm. \tag{10}$$

In the above, C_1 and C_2 are arbitrary constants. It is straightforward to check that the Jacobi identity is preserved. When $C_2 = 0$ and $C_1 = 2$ or $C_1 = -2$, we have the $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$ algebra, respectively. We will be interested in the cubic algebra that is treated as a deformation of the $\mathfrak{su}(2)$ algebra. Hence, we will consider finite-dimensional representations only.

The finite-dimensional irreducible representations of the HA are characterized, like in $SU(2)$, by an integer or half-integer j of dimension $2j+1$.

$$Z|j,m\rangle = m|j,m\rangle, \qquad X_+|j,m\rangle = \sqrt{g(j) - g(m)}|j,m+1\rangle. \tag{11}$$

The structure function $g(Z)$ is chosen such that $f(Z) = g(Z) - g(Z-1)$. Note that $g(Z)$ is defined only up to the addition of a constant. Later we will see that this freedom plays an important role in arriving at the one parameter family of surfaces, namely equation (3). For arbitrary polynomials $f(Z)$, one can solve and find solutions for $g(Z)$ [27]. The fact that we can write $f(Z)$ as a difference of structure functions $g(Z)$ enables us to find the Casimir $\mathcal{C}$ of the algebra in an almost trivial way. The Casimir $\mathcal{C}$ is

$$\mathcal{C} = \tfrac{1}{2}[\{X_+, X_-\} + g(Z) + g(Z-1)], \tag{12}$$

$$\mathcal{C}|j,m\rangle = g(j)|j,m\rangle, \tag{13}$$

where the curly brackets denote anti-commutator. It is easy to verify $[\mathcal{C}, X_\pm] = [\mathcal{C}, Z] = 0$. So far we have not specified what the explicit form of $g(Z)$ is and without further ado we state for our case of HA:

$$g(Z) = C_0 + \frac{C_1}{2} Z(Z+1) + \frac{C_2}{4} Z^2(Z+1)^2. \tag{14}$$

Here, C_0 is a constant. Now the Casimir as a function of z alone assumes a form of a single- or double-well potential depending on the values of the parameters. The physical meaning of this behavior can be understood from the work of Rocek [26]. The condition for finite-dimensional representations is also discussed in [26]. In our case we note that $g(Z) = g(-Z-1)$, which is also the condition for the case of the $SU(2)$ algebra. This makes the function g periodic and hence we can be sure that we have finite-dimensional representations for the choice of parameters we will make for our Higgs algebra.

Applying equation (14) to equation (12) and then comparing it with equation (3), we get $C_0 = \mu^2$, $C_1 = -2(2\mu+1)$ and $C_2 = 4$. Let us note that though there is a singularity in the continuum limit, in the discrete case we have a valid representation theory as we vary the parameters. This looks like a novel resolution of singularity. Similar behavior was noted in [23] where the topology changes from a sphere to a torus with a degenerate surface at a transition point in the parameter space.

Now we will construct the CS for this nonlinear algebra to get a better understanding of the semiclassical behavior.

4. The Higgs algebra coherent states

The field CS [31, 32] and their generalizations [33–35] have been extensively studied from various aspects, motivated mainly by applications to quantum optics. But, we are interested in them as providing appropriate semiclassical descriptions of the nonlinear algebra. As is well known there are two types of CS: (1) those that are 'annihilation operator' eigenstates also known as Barut–Girardello CS [33], (2) states obtained through the action of the displacement operator also known as Perelemov states [35]. The first is useful when considering non-compact groups like $SU(1, 1)$ and the second for compact ones.

We consider the finite-dimensional representation of the Higgs algebra as we want to view it as a deformation of the fuzzy sphere algebra. Hence, we resort to the construction via the displacement operator. One should keep in mind that since our algebra is nonlinear one cannot attach any group theoretical interpretation to such states. The actual procedure should be viewed as an algebraic construction and has been carried out in [36, 37].

Since the algebra under study is not a Lie algebra, a straightforward application of the Perelomov prescription is also not possible, wherein essential use of the Baker–Campbell–Hausdorff (BCH) formula is made. To overcome this we find a new operator $\widehat{X}_-$ such that $[X_+, \widehat{X}_-] = 2\,Z$. Let $\widehat{X}_- = X_-\,G(\mathcal{C}, Z)$; substituting this in the commutator relation we get

$$X_+X_-G(\mathcal{C}, Z) - X_-X_+G(\mathcal{C}, Z+1) = 2Z. \tag{15}$$

Choose the ansatz for G of the form

$$G(\mathcal{C}, Z) = \frac{-Z(Z+1)+\lambda}{\mathcal{C} - g(Z-1)}, \tag{16}$$

where λ is an arbitrary constant. Now that we have the 'ladder' operators that obey the $\mathfrak{su}(2)$ algebra, we can use the Perelomov prescription. The CS are given by

$$|\zeta\rangle = \mathrm{e}^{\zeta X_+ - \zeta^* \widehat{X}_-}|j, -j\rangle. \tag{17}$$

Disentangling the above exponential, using the BCH formula for $\mathfrak{su}(2)$ and $\widehat{X}_-|j, -j\rangle = 0$, we find that the expression for the CS acquires the form

$$|\zeta\rangle = N^{-1}(|\zeta|^2)\,\mathrm{e}^{\zeta X_+}|j, -j\rangle, \tag{18}$$

where $N^{-1}(|\zeta|^2)$ is the normalization constant that is yet to be determined and $\zeta \in \mathbb{C}$. Note that the ladder operators that form the 'Lie algebra' are not mutually adjoint. The above state is to be viewed as a 'nonlinear $\mathfrak{su}(2)$ coherent state' and is very similar in properties to the CS of nonlinear oscillators [38] and extensively used in quantum optics.

Now we will study whether the above definition of CS is suitable. The requirements for $|\zeta\rangle$ to be CS have been enunciated by Klauder [39]: (1) $|\zeta\rangle$ should be normalizable, (2) $|\zeta\rangle$ should be continuous in ζ and (3) $|\zeta\rangle$ should satisfy the resolution of identity. We will consider the normalization and resolution of identity in the following.

4.1. Normalization

To find the normalization constant $N^{-1}(|\zeta|^2)$ we compute the scalar product of HACS and set it equal to 1. We get

$$\begin{aligned} N^2(|\zeta|^2) &= \langle j, -j|\,\mathrm{e}^{\bar{\zeta} X_-}\,\mathrm{e}^{\zeta X_+}|j, -j\rangle, \\ &= 1 + \sum_{n=1}^{2j} \frac{|\zeta|^{2n}}{(n!)^2} \prod_{\ell=0}^{n-1} K_{j,-j+\ell} \prod_{p=0}^{n-1} H_{j,-j+n-p}, \end{aligned}$$

$$= 1 + \sum_{n=1}^{2j} \frac{|\zeta|^{2n}}{(n!)^2} \prod_{\ell=0}^{n-1} (K_{j,-j+\ell})^2,$$

$$= 1 + \sum_{n=1}^{2j} |\zeta|^{2n} \binom{2j}{n}$$

$$\times \prod_{\ell=0}^{n-1} \left(\frac{C_1}{2} + \frac{C_2}{4} [2j(j-\ell) - \ell(\ell+1)] \right). \tag{19}$$

In the above expression $K_{j,m} \equiv H_{j,m+1} = \sqrt{g(j) - g(m)}$. Observe that the expression under the product is quadratic in ℓ and can be factorized:

$$N^2(|\zeta|^2) = 1 + \sum_{n=1}^{2j} |\zeta|^{2n} \binom{2j}{n} \prod_{\ell=0}^{n-1} (\ell - A_+)(\ell - A_-),$$

$$= 1 + \sum_{n=1}^{2j} |\zeta|^{2n} D_n, \tag{20}$$

where

$$A_\pm = -\left[\left(j + \frac{1}{2} \right) \pm \sqrt{\left(j + \frac{1}{2} \right)^2 + \left(2j^2 + \frac{2C_1}{C_2} \right)} \right]. \tag{21}$$

Taking the ratio of D_{n+1}/D_n we get

$$\frac{D_{n+1}}{D_n} = \frac{(n - A_+)(n - A_-)(2j - n)}{(n+1)}. \tag{22}$$

It can be seen that this is the condition for the generalized hypergeometric series for ${}_3F_0(-A_+, -A_-, -2j; 0; -|\zeta|^2)$. Hence, the final expression for the normalization constant of the HA is

$$N^2(|\zeta|^2) = {}_3F_0(-A_+, -A_-, -2j; 0; -|\zeta|^2). \tag{23}$$

4.2. Resolution of identity

The resolution of identity is a very important criterion that any CS must satisfy:

$$\frac{1}{\pi} \int |\zeta\rangle \, d\mu(\zeta, \bar{\zeta}) \langle\zeta| = \mathbb{I}. \tag{24}$$

The integration is over the complex plane. Introducing the HACS in the above equation and writing the resulting equation in terms of angular coordinates, $\zeta = r\,e^{i\theta} (0 \leqslant \theta < 2\pi)$, brings us to

$$\sum_{n=0}^{2j} \int dr \frac{\rho(r^2)}{N^2(r^2)} \frac{r^{2n+1}}{(n!)^2} X_+^n |j, -j\rangle\langle -j, j| X_-^n = \mathbb{I}. \tag{25}$$

We know that the angular momentum states are complete and hence for the above equality to hold the integral should be equal to 1. Defining $\tilde{\rho}(r^2) \equiv \rho(r^2)/N^2(r^2)$ and simplifying the product as shown in the previous subsection we have

$$\int_0^\infty dr r^{2n+1} \tilde{\rho}(r^2) = \Gamma(n+1) \frac{\Gamma(A'_+ - n + 1)\Gamma(A'_- - n + 1)\Gamma(2j - n + 1)}{\Gamma(2j+1)\Gamma(A'_+ + 1)\Gamma(A'_- + 1)}, \tag{26}$$

where $A'_{\pm} = -A_{\pm}$. Making a change of variable, $r^2 = x$, and replacing the discrete variable n by the complex one $(s-1)$ we note that the weight function $\tilde{\rho}(x)$ and the rhs of the above equation become a Mellin transform-related pair [40]. The unknown function $\tilde{\rho}(x)$ can be read off from tables of Mellin transforms [41]. For the sake of completeness we reproduce the relevant formula below:

$$\int_0^\infty \mathrm{d}x x^{s-1}\, G^{m,n}_{p,q}\left(x \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q, 0 \end{matrix}\right) = \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)}, \tag{27}$$

where the rhs is the s-dependent part of the weight function. $G^{p,\,q}_{m,\,n}$ is called the Meijer-G function and more details can be found in [42]. Casting equation (26) in the above standard form we find that

$$\rho(|\zeta|^2) = \frac{{}_3F_0(A'_+, A'_-, -2j; 0; -|\zeta|^2)}{\Gamma(2j+1)\Gamma(A'_+ + 1)\Gamma(A'_- + 1)} G^{1,3}_{0,0}\left[-|\zeta|^2 \middle| \begin{matrix} -2j-1, -(A'_+ + 1), -(A'_- + 1) \\ 0 \end{matrix}\right]. \tag{28}$$

At $\mu = 1$, this function is well behaved, leaving no trace of the conical singularity encountered in the continuum. This can be seen as a consequence of quantizing the Higgs manifold using the HACS.

5. The star product

The star product plays a 'stellar' role in the study of deformation quantization and non-commutative geometry. There exist in the literature a host of such products depending on various situations. For example the Moyal product arises when one is dealing with a non-commutative plane and the underlying algebra happens to be of the Heisenberg–Weyl type. Similarly the Kontsevich product arises when the non-commutative parameter itself is a function of the coordinates. We in this section are interested in constructing a new product that will reduce to the star product of the fuzzy sphere when the parameter pertaining to the cubic term is set to zero. The technique for obtaining the star product for the HA follows Grosse and Presnajder [43] and we refer to it for details regarding the use of CS in this construction. It suffices to mention here that CS ensures that the product obtained is associative.

The algebra of functions on the Higgs manifold, $\mathcal{M}_H$, is commutative under point-wise multiplication. When we quantize this manifold, this point-wise product is deformed to an associative star product which is non-commutative.

We consider the algebra of operators, $\mathcal{A}$, generators of which satisfy the HA. These operators act on some Hilbert space. We then use the symbol to map these operators to the functions on the Higgs manifold. The symbol map is defined as follows:

$$\phi : \mathcal{A} \to \mathcal{M}_H. \tag{29}$$

We use the HACS to define the symbol map in this case: $\phi(\hat{\alpha}) \equiv \langle\zeta|\, \mathrm{e}^{\alpha_- X_-}\, \mathrm{e}^{\alpha_0 Z}\, \mathrm{e}^{\alpha_+ X_+}|\zeta\rangle$,

$$\begin{aligned} \phi(\hat{\alpha}) &= N^{-2}\langle j, -j|\, \mathrm{e}^{\alpha_0 Z}\, \mathrm{e}^{-\alpha_0 Z}\, \mathrm{e}^{(\alpha_- + \zeta^*)X_-}\, \mathrm{e}^{\alpha_0 Z}\, \mathrm{e}^{(\alpha_+ + \zeta)X_+}|j, -j\rangle, \\ &= N^{-2}\, \mathrm{e}^{-j\alpha_0}\langle j, -j|\, \mathrm{e}^{(\alpha_- + \zeta^*)\,\mathrm{e}^{\alpha_0} X_-}\, \mathrm{e}^{(\alpha_+ + \zeta)X_+}|j, -j\rangle, \\ &= \mathrm{e}^{-j\alpha_0}\frac{{}_3F_0(-A_+, -A_-, -2j; 0; -(\alpha_+ + \zeta)(\alpha_- + \zeta^*)\,\mathrm{e}^{\alpha_0})}{{}_3F_0(-A_+, -A_-, -2j; 0; -|\zeta|^2)}, \end{aligned} \tag{30}$$

where A_+ and A_- are as defined in section 4. In the above derivation we have made use of the identity

$$\mathrm{e}^{\alpha Z} X_+\, \mathrm{e}^{-\alpha Z} = \mathrm{e}^{\alpha} X_+. \tag{31}$$

We will use this identity in simplifying the symbol of the product of two general operators labeled by $\hat{\alpha}$ and $\hat{\beta}$.

We now compute the following to find the star product in terms of deformations of the point-wise product:

$$\phi(\hat{\alpha}\hat{\beta}) = N^{-2}\langle\zeta|\, \mathrm{e}^{\alpha_- X_-}\, \mathrm{e}^{\alpha_0 Z}\, \mathrm{e}^{\alpha_+ X_+}\, \mathrm{e}^{\beta_- X_-}\, \mathrm{e}^{\beta_0 Z}\, \mathrm{e}^{\beta_+ X_+}|\zeta\rangle. \tag{32}$$

We give the final result of this matrix element without going through the steps:

$$\begin{aligned}\phi(\hat{\alpha}\hat{\beta}) = {} & N^{-2}\phi(\hat{\alpha})\phi(\hat{\beta}) + N^{-2}\,\mathrm{e}^{-j(\alpha_0+\beta_0)}[\phi(\hat{\alpha})\,\mathrm{e}^{j\alpha_0} + \chi(\hat{\alpha})\,\mathrm{e}^{j\alpha_0} \\ & + \mathrm{e}^{j(\alpha_0+\beta_0)}\{\phi(\hat{\alpha})\phi(\hat{\beta}) + \chi(\hat{\alpha})\chi(\hat{\beta}) + \phi(\hat{\alpha})\chi(\hat{\beta}) + \chi(\hat{\alpha})\phi(\hat{\beta})\} + L].\end{aligned} \tag{33}$$

In this expression

$$\chi(\hat{\alpha}) = \mathrm{e}^{-j\alpha_0}\sum_{i=1}^{2j}\frac{(\alpha_- + \zeta^*)^i(-\zeta)^i\,\mathrm{e}^{\mathrm{i}\alpha_0}}{(i!)^2}\prod_{l=0}^{i-1}K^2_{j,-j+l}\sum_{k=0}^{i-1}\frac{(\alpha_+ + \zeta)^k}{(-\zeta)^k}\binom{i}{k}, \tag{34}$$

$$\chi(\hat{\beta}) = \mathrm{e}^{-j\beta_0}\sum_{i=1}^{2j}\frac{(\beta_+ + \zeta)^i(-\zeta^*)^i\,\mathrm{e}^{\mathrm{i}\beta_0}}{(i!)^2}\prod_{l=0}^{i-1}K^2_{j,-j+l}\sum_{k=0}^{i-1}\frac{(\beta_- + \zeta^*)^k}{(-\zeta^*)^k}\binom{i}{k} \tag{35}$$

and

$$\begin{aligned}L = {} & \sum_{i=1}^{2j}\frac{(\beta_+ + \zeta)^i(\alpha_- + \zeta^*)^i\,\mathrm{e}^{\mathrm{i}(\alpha_0+\beta_0)}}{(i!)^2}\prod_{l=0}^{i-1}K^2_{j,-j+l} \\ & \times\Bigg[{}_3F_0(-A_+, -A_-, i-2j; 0; -\alpha_+(\alpha_- + \zeta^*)\,\mathrm{e}^{\alpha_0}) + \sum_{m=1}^{i-1}\frac{\beta_-^m}{(\alpha_- + \zeta^*)^m\,\mathrm{e}^{m\alpha_0}}\binom{i}{m} \\ & \times\, {}_3F_0(-A_+, -A_-, i-2j-m; 0; -\alpha_+(\alpha_- + \zeta^*)\,\mathrm{e}^{\alpha_0})\Bigg] + 1.\end{aligned} \tag{36}$$

The computations involve some non-trivial simplifications to bring it to equation (33).

We see that the first term in equation (33) is the point-wise product of the two symbols and the term in the bracket gives the deformations.

As the star product was computed using the HACS we can be sure that they are well behaved at the conical singularity seen in the continuum. The reason is the same as mentioned in section 3.

6. Conclusions

In the present paper we have analyzed algebraic structures that are more general than the fuzzy sphere. We have shown that the interesting feature of the topology change studied in [23] is also present in the present case. It must be mentioned here that the HA arises naturally in the study of integrable dynamics of two-dimensional curved surfaces. The study of nonlinear deformations of Lie algebra and their representation theory is important in the context of non-commutative geometries. There have been attempts to write down the Dirac operator for the $SU_q(2)$ algebra [44, 45]. Such attempts can be extended to this nonlinear algebra too. Quantum field theories on fuzzy spheres coming from $SU(2)$ representations can be further extended to our framework also. The topology change will play an important role in such studies which will be explored in detail. Fuzzy spheres are considered in the Kaluza–Klein framework and similar studies for the Higgs algebra will bring out new features. These novel surfaces can also be generated dynamically along the lines of fuzzy spheres and they will naturally arise when one introduces higher dimensional operators in the effective action of QFTs.

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